1 Households' Problem

Euler Equation. Denote p_t the wealth / consumption ratio. Denote μ_{pt} and σ_{pt} the geometric drift and volatility of p_t. Denote μ_{ct} and σ_{ct} the geometric drift and volatility of the consumption process of the representative agent.

The Euler equation in continuous-time is

$$\sigma_{ct} = \frac{\kappa_t}{\gamma} + \frac{1-\gamma\psi}{\gamma(\psi-1)}\sigma_{pt}$$
$$\mu_{ct} = \psi(r_t - \rho) + \frac{1+\psi}{2\gamma}\kappa_t^2 + \frac{1-\gamma\psi}{\gamma(\psi-1)}\kappa_t\sigma_{pt} - \frac{1-\gamma\psi}{2(\psi-1)\gamma}\sigma_{pt}^2$$

Market Pricing. We have

$$0 = \frac{1}{p_t} + \mu_{ct} + \mu_{pt} - r_t - \kappa_t(\sigma_{ct} + \sigma_{pt})$$

Substituting r_t and κ_t using the Euler equation, we obtain

$$0 = \frac{1}{p_t} - \rho + (1 - \frac{1}{\psi})(\mu_{ct} - \frac{1}{2}\gamma\sigma_{ct}^2) + \mu_{pt} + (1-\gamma)\sigma_{pt}'\sigma_{Ct} + \frac{1}{2}\frac{\frac{1}{\psi} - \gamma}{1 - \frac{1}{\psi}}\sigma_{pt}^2 \quad (1)$$

2 Long run risk model

Assume the following process for aggregate consumption:

$$\frac{dC_t}{C_t} = \mu_t dt + \nu_D\sqrt{\sigma_t}dZ_t$$
$$d\mu_t = \kappa_\mu(\bar{\mu} - \mu_t)dt + \nu_\mu\sqrt{\sigma_t}dZ_t^\mu$$
$$d\sigma_t = \kappa_\sigma(1 - \sigma_t)dt + \nu_\sigma dZ_t^\sigma$$

Write $p_t = p(\mu_t, \sigma_t)$. Applying Ito's lemma we have

$$\mu_{pt} = (\bar{\mu} - \mu_t)\frac{\partial_\mu G}{G}\kappa_\mu + \frac{1}{2}\nu_\mu^2\sigma_t\frac{\partial_\mu^2 G}{G} + \kappa_\sigma(1-\sigma_t)\frac{\partial_\sigma G}{G} + \frac{1}{2}\nu_\sigma^2\frac{\partial_\sigma^2 G}{G}$$
$$\sigma_{pt} = \begin{pmatrix} 0 \\ \nu_\mu\sqrt{\sigma_t}\frac{\partial_\mu G}{G} \\ \nu_\sigma\frac{\partial_\sigma G}{G} \end{pmatrix}$$

Plugging these expressions for μ_{pt} and σ_{pt} into Equation (1), we obtain the following PDE for p_t:

$$\begin{aligned} 0 &= \frac{1}{p} - \rho + (1 - \frac{1}{\psi})(\mu - \frac{1}{2}\gamma\nu_D^2\sigma) \\ &+ \kappa_\mu(\bar{\mu} - \mu)\frac{\partial_\mu G}{G} + \frac{1}{2}\nu_\mu^2\sigma\frac{\partial_\mu^2 G}{G} + \kappa_\sigma(1-\sigma)\frac{\partial_\sigma G}{G} + \frac{1}{2}\nu_\sigma^2\frac{\partial_\sigma^2 G}{G} \\ &+ \frac{1}{2}\frac{\frac{1}{\psi} - \gamma}{1 - \frac{1}{\psi}}(\nu_\mu^2\sigma(\frac{\partial_\mu G}{G})^2 + \nu_\sigma^2(\frac{\partial_\mu G}{G})^2) \end{aligned}$$

3 Comparison

Correspondences	
$\Delta c_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}$	$\frac{dC_t}{C_t} = \mu_t dt + \nu_D \sqrt{\sigma_t} dZ_t$
$x_{t+1} = \rho x_t + \phi_e \sigma_t e_{t+1}$	$d\mu_t = \kappa_\mu (\bar{\mu} - \mu_t) dt + \nu_\mu \sqrt{\sigma_t} dZ_t^\mu$
$\sigma_{t+1}^2 = \sigma^2 + \nu_1 (\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}$	$d\sigma_t = \kappa_\sigma (1 - \sigma_t) dt + \nu_\sigma dZ_t^\sigma$

Correspondences		
growth rate	μ	$\bar{\mu} = \mu \times 12$
volatility	σ^2	$\nu_D = \sigma \times \sqrt{12}$
growth persistence	ρ	$\kappa_\mu = (1 - \rho) \times 12$
volatility persistence	ν_1	$\kappa_\sigma = (1 - \nu_1) \times 12$
growth rate volatility	φ_e	$\nu_\mu = (\varphi_e \sigma 12) \times \sqrt{12}$
volatility volatility	σ_w	$\nu_\sigma = (\sigma_w / \overline{\sigma}^2) \times \sqrt{12}$
time discount	δ	$\rho = (1 - \delta) \times 12$

BY 2004		
Name	Discrete Time	Continuous Time
growth rate	$\mu = 0.0015$	$\overline{\mu} = 0.018$
volatility	$\sigma = 0.0078$	$\nu_D = 0.027$
growth persistence	$\rho = 0.979$	$\kappa_\mu = 0.252$
volatility persistence	$\nu_1 = 0.987$	$\kappa_\sigma = 0.156$
growth rate volatility	$\phi_e = 0.044$	$\nu_\mu = 0.0143$
volatility volatility	$\sigma_w = 0.0000023$	$\nu_\sigma = 0.131$
time discount	$\delta = 0.998$	$\rho = 0.024$

BKY 2007		
Name	Discrete Time	Continuous Time
mean growth rate	$\mu = 0.0015$	$\overline{\mu} = 0.018$
mean volatility	$\sigma = 0.0072$	$\nu_D = 0.025$
growth persistence	$\rho = 0.975$	$\kappa_\mu = 0.3$
volatility persistence	$\nu_1 = 0.999$	$\kappa_\sigma = 0.012$
growth rate volatility	$\phi_e = 0.038$	$\nu_\mu = 0.0114$
volatility volatility	$\sigma_w = 0.00000283$	$\nu_\sigma = 0.189$
time discount	$\delta = 0.9989$	$\rho = 0.0132$