1 Pseudo- visco- acoustic VTI variable density coupled second order self adjoint system

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2 Introduction

This note shows the derivation of *time update equations* for the pseudo- visco- acoustic vertical transverse isotropy (VTI) variable density coupled second order self-adjoint system. We implement attenuation with a monochromatic approximation to Maxwell bodies, and use this attenuation model to implement zero outgoing absorbing boundary conditions on the exterior of the modeling domain.

The time update equations are used to advance solutions in time, expressing the quasi-p pressure wavefield $P_{(t+\Delta)}$ and quasi-s pressure wavefield $M_{(t+\Delta)}$ at the next time step $(t+\Delta)$ as functions of $P_{(t-\Delta)}, P_{(t)}$ and $M_{(t-\Delta)}, P_{(t)}$

2.1 Symbols

$\partial_t,\ \partial_x,\ \partial_y,\ \partial_z$	$\dfrac{\partial}{\partial t},\ \dfrac{\partial}{\partial x},\ \dfrac{\partial}{\partial y},\ \dfrac{\partial}{\partial z}$
Δ	Temporal sampling
ω, Q	reference frequency for attentuation, attenuation at frequency ω
P, M	quasi-P, quasi-S wavefields
$s_p(x,y,z,t),\ s_m(x,y,z,t)$	quasi-P, quasi-S source terms
f	$1 - \dfrac{V_p^2}{V_s^2}$
$\widehat{\eta}$	$\sqrt{\dfrac{2\,(\epsilon-\delta)}{f+2\epsilon}}$
b	buoyancy $= 1/\rho$ (reciprocal density)
$\{\ V_p,\ \epsilon,\ \widehat{\eta}\ \}$	Material parameters

2.2 Coupled second order modeling system

Equation 1 shows the modeling system with absorbing boundaries implemented using amplitude only (dissipation only, no dispersion) Q.

We apply the time derivatives in the terms $\frac{\omega}{Q}\partial_t p$ and $\frac{\omega}{Q}\partial_t m$ using a backward one-sided numerical difference. We tested both forward one-sided and centered difference alternatives and found them to be less stable.

$$\begin{aligned}
\frac{b}{V_p^2}\left(\partial_t^2(P) + \frac{\omega}{Q}\partial_t(P)\right) &= \partial_x\left(b(1+2\epsilon)\partial_x(P)\right) + \partial_y\left(b(1+2\epsilon)\partial_y(P)\right) + \partial_z\left(b(1-f\widehat{\eta}^2)\partial_z(P)\right) + \partial_z\left(bf\widehat{\eta}\sqrt{1-\widehat{\eta}^2}\partial_z(M)\right) + s_p \\
\frac{b}{V_p^2}\left(\partial_t^2(M) + \frac{\omega}{Q}\partial_t(M)\right) &= \partial_x\left(b(1-f)\partial_x(M)\right) + \partial_y\left(b(1-f)\partial_y(M)\right) + \partial_z\left(b(1-f+f\widehat{\eta}^2)\partial_z(M)\right) + \partial_z\left(bf\widehat{\eta}\sqrt{1-\widehat{\eta}^2}\partial_z(P)\right) + s_m
\end{aligned} \tag{1}$$

3 Time update equations

3.1 Time update numerical difference formulas, first and second order

$$\partial_t p = \frac{1}{\Delta}\left[p_{(t)} - p_{(t-\Delta)}\right] \tag{2}$$

$$\begin{aligned}
\partial_t^2 p &= \frac{1}{\Delta^2}\left[p_{(t+\Delta)} - 2p_{(t)} + p_{(t-\Delta)}\right] \\
p_{(t+\Delta)} &= \Delta^2\partial_t^2 p + 2p_{(t)} - p_{(t-\Delta)}
\end{aligned} \tag{3}$$

3.2 Rearrange terms for $\partial_t^2(P)$ and $\partial_t^2(M)$

$$\partial_t^2(P) = \frac{V_p^2}{b}\left[\partial_x\left(b(1+2\epsilon)\partial_x(P)\right) + \partial_y\left(b(1+2\epsilon)\partial_y(P)\right) + \partial_z\left(b(1-f\widehat{\eta}^2)\partial_z(P)\right) + \partial_z\left(bf\widehat{\eta}\sqrt{1-\widehat{\eta}^2}\partial_z(M)\right) + s_p\right] - \frac{\omega}{Q}\partial_t(P)$$

$$\partial_t^2(M) = \frac{V_p^2}{b}\left[\partial_x\left(b(1-f)\partial_x(M)\right) + \partial_y\left(b(1-f)\partial_y(M)\right) + \partial_z\left(b(1-f+f\widehat{\eta}^2)\partial_z(M)\right) + \partial_z\left(bf\widehat{\eta}\sqrt{1-\widehat{\eta}^2}\partial_z(P)\right) + s_m\right] - \frac{\omega}{Q}\partial_t(M)$$

3.3 Apply equations 2 and 3, and rearrange

$$
\begin{aligned}
P^{t+\Delta} = {} & \Delta^2\ \frac{V_p^2}{b}\left[\partial_x b\left((1+2\epsilon)\partial_x(P)\right) + \partial_y b\left((1+2\epsilon)\partial_y(P)\right) + \partial_z b\left((1-f\widehat{\eta}^2)\partial_z(P)\right) + \partial_z b\left(f\widehat{\eta}\sqrt{1-\widehat{\eta}^2}\partial_z(M)\right) + s_p\right] \\
& - \Delta\ \frac{\omega}{Q}\left(P^t - P^{t-\Delta}\right) + 2P^t - P^{t-\Delta}
\end{aligned}
$$

$$
\begin{aligned}
M^{t+\Delta} = {} & \Delta^2\ \frac{V_p^2}{b}\left[\partial_x b\left((1-f)\partial_x(M)\right) + \partial_y b\left((1-f)\partial_y(M)\right) + \partial_z b\left((1-f+f\widehat{\eta}^2)\partial_z(M)\right) + \partial_z b\left(f\widehat{\eta}\sqrt{1-\widehat{\eta}^2}\partial_z(P)\right) + s_m\right] \\
& - \Delta\ \frac{\omega}{Q}\left(M^t - M^{t-\Delta}\right) + 2M^t - M^{t-\Delta}
\end{aligned}
$$